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Effective One-Dimensional Equation of Motion for Nuclear Fission

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An approach for describing the dynamics of nuclear fission in the framework of generalized quantum mechanics is discussed. The collective kinetic energy is assumed to be two dimensional, and the reduced mass is allowed to vary with the coordinates. The generalized calculus of variation is employed for minimizing the action after being properly quantized as required by Hamilton's principle, employing a curvilinear coordinate system. The corresponding Euler Lagrange equation is identified as the required generalized equation of motion. The proposed generalized two-dimensional equation of motion is separated into a vibrational eigenvalue equation and a set of coupled-channel one-dimensional equations which describe the translational motion, by exploiting the completeness of the vibrational eigenfunctions. Such a system of coupled equations can be decoupled by replacing the coupling matrix elements by a nonlocal interaction, which can be rendered local after employing the effective mass approximation. As a consequence this differential equation is provided with an effective mass, an effective potential barrier, and a differential boundary term which is responsible for restoring the self-adjointness of the kinetic energy differential operator.

1. INTRODUCTION

In view of the fact that the existence of dips in the potential fission barrier is well established both theoretically (Strutinsky, 1967, 1968) and experimentally (Polikonov, 1962), the present paper develops a simplified quantal treatment for describing the origin and consequence of such energy dips on the fission dynamics.

An attempt was made by Morsy *et al.* (1979) to develop a simplified quantal treatment for describing the dynamics of nuclear fission on the basis

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of an optical model approximation and in terms of a double-centered Wood–Saxon potential barrier.

However, the theoretical treatments for describing the problem of nuclear fission do not guarantee self-adjointness of the kinetic energy differential operator (Bloch, 1957; Lane and Robson, 1966; Morsy and Ata, 1971a–c).

The main objective of this paper is to establish a generalized quantal treatment for nuclear fission that is free from the above-mentioned shortcoming. This can be accomplished by:

(1) Establishing in terms of curvilinear coordinates a generalized two-dimensional scattering equation, by employing the generalized calculus of variation in the framework of Hamilton's principle after being properly quantized. The kinetic energy differential operator is not only self-adjoint by itself, but also is provided with a varying reduced mass.

(2) Separating out the generalized two-dimensional scattering equation into a vibrational eigenvalue equation and a set of coupled-channel one-dimensional translation equations which is provided with varying reduced mass, and a differential boundary term.

(3) Decoupling the set of coupled-channel equations by applying the optical model and effective mass approximations to a differential equation provided with an effective fission barrier (EFB), an effective reduced mass (ERM), and a differential boundary term that is responsible for making the kinetic energy to be intrinsically self-adjoint. In addition, both the EFB and ERM satisfy the appropriate asymptotic requirements.

2. GENERALIZED EULER–LAGRANGE DIFFERENTIAL EQUATION

Let us assume that the classical kinetic energy of collective nuclear motion is bilinear in velocities with the form (Morsy *et al.*, 1979)

$$T = \tfrac{1}{2}\nu(u)[\eta^2(v)\dot{u}^2 + \dot{v}^2] \tag{2.1}$$

in which u, v, and $\nu(u)$ correspond, respectively, to motion along the fission path ($v = 0$) and perpendicular to it, with the varying reduced mass of the system

$$\eta(v) = 1 + vk_0\theta(u_0 - |u|) \tag{2.2}$$

where k_0 refers to the constant curvature of the fission coordinate, $\theta(x)$ is the usual step function, and the choice of u_0 and k_0 is made such that the displacement of the fission coordinates $v = 0$ from the path of minimum

potential energy is kept minimum. As a consequence, the corresponding Lagrangian function reads

$$L = \frac{1}{2}\left[\frac{p_u^2}{\nu(u)\eta^2(v)} + \frac{p_v^2}{\nu(u)}\right] - V(u, v) \tag{2.3}$$

where $V(u, v)$ denotes the potential energy function. This Lagrangian can be written more formally in terms of the momenta p_u, p_v as

$$L(u, v, p_u, p_v) = \frac{1}{2}\left(p_u^T \frac{1}{\nu\eta^2} p_u + p_v^T \frac{1}{\nu} p_v\right) - V(u, v) \tag{2.4}$$

where p_x and p_x^T stand for the momentum and its transpose, respectively, and ν and η^2 stand for $\nu(u)$ and $\eta^2(u)$, respectively.

At this point, we are in a position to incorporate the quantal features in the above classical Lagrangian. This can be accomplished by following the quantization procedure developed by Morsy *et al.* (1982), Embaby (1978), Morsy and Embaby (1986a, b), and El-Sabagh (1979). This procedure requires the replacement of every momentum variable P_x by the corresponding differential operator $\hat{P}_x$, namely

$$P_x \to \hat{P}_x = \frac{\hbar}{i}\frac{\vec{\partial}}{\partial x} \equiv \frac{\hbar}{i} D_x \tag{2.5}$$

and

$$P_x^T \to \hat{P}_x^T = \frac{\hbar}{i}\frac{\vec{\partial}}{\partial x} \equiv \frac{\hbar}{i} \complement_x \tag{2.6}$$

As a result, the quantal analogue of the Lagrangian can be represented as

$$\hat{L}(u, v, p_u, p_v) = -\complement_u \frac{\hbar^2}{2\nu\eta^2} D_u - \complement_v \frac{\hbar^2}{2\nu} D_v - V(u, v) \tag{2.7}$$

in which

$$D_x = \frac{\vec{\partial}}{\partial x} \qquad \text{and} \qquad \complement_x = \frac{\overleftarrow{\partial}}{\partial x}$$

In terms of curvilinear coordinates (u, v) the quantal analogue $\langle\hat{L}\rangle$ is simply the expectation value of $\hat{L}$ (Morsy and Ata, 1971a–c; Morsy *et al.*, 1982)

$$\langle\hat{L}\rangle = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \sqrt{g}\, \psi^*(u, v, t)\hat{L}(u, v, P_u, P_v)\psi(u, v, t) \tag{2.8}$$

where g denotes the determinant

$$g = \begin{bmatrix} \eta^2(v)\nu(u) & 0 \\ 0 & \nu(u) \end{bmatrix} = \eta^2(v)\nu^2(u) \tag{2.9}$$

where $\psi(u, v, t)$ denotes the state function of the system, whose modulus squared gives the probability density; one expects that the total probability would be

$$\int_{t_0}^{t_1} dt \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \sqrt{g}|\psi(u, v, t)|^2 = 1 \tag{2.10}$$

Now, by virtue of equations (2.7) and (2.8), we get

$$\langle \hat{L} \rangle =$$

$$\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \sqrt{g}\psi^*(u, v, t)\left[-\mathfrak{A}_u \frac{\hbar^2}{2\nu\eta} D_u - \mathfrak{A}_v \frac{\hbar^2}{2\nu} D_v - V(u, v)\right]\psi(u, v, t) \tag{2.11}$$

The Hamilton principle of least action requires that the functional variation of the quantal action

$$A_q = \int_{t_0}^{t_1} dt \, \langle \hat{L} \rangle \tag{2.12}$$

must vanish identically for every choice of the variational variable $\delta\psi(u, v, t)$ subject to the constrained integral given by (2.10), namely

$$\delta A_q = \delta \int_{t_0}^{t_1} dt \, \langle \hat{L} \rangle = 0 \tag{2.13}$$

Such a constrained variational problem can be reduced to an unconstrained one by introducing a Lagrangian multiplier λ defined as (Morsy and Embaby, 1986)

$$\lambda\left[\int_{t_0}^{t_1} dt \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \sqrt{g} \, |\psi(u, v, t)|^2 - 1 \right] = 0 \tag{2.14}$$

As a consequence, the quantal action becomes

$$A_q = \int_{t_0}^{t_1} dt \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \sqrt{g} \, \psi^*\left[-\mathfrak{A}_u \frac{\hbar^2}{2\nu\eta} D_u - \mathfrak{A}_v \frac{\hbar^2}{2\nu} D_v - V(u, V)\right]\psi$$

$$+ \lambda\left[\int_{t_0}^{t_1} dt \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \sqrt{g} \, |\psi|^2 - 1 \right] \tag{2.15}$$

Clearly, the condition

$$\delta \int_{t_0}^{t_1} dt \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \sqrt{g}\, F[\psi, \psi'_u, \psi'_v, \psi^*, \psi'^*_u, \psi'^*_v) = 0 \quad (2.16)$$

in which

$$F(\psi, \psi'_u, \psi'_v, \psi^*, \psi'^*_u, \psi'^*_v) = -\frac{\hbar^2}{2v\eta^2}|\psi'_u|^2 - \frac{\hbar^2}{2v}|\psi'_v|^2 - (V-\lambda)|\psi|^2 - \frac{1}{\sqrt{g}}\psi^*\left(\frac{\partial\sqrt{g}}{\partial u}\right)\left(\frac{\hbar^2}{2v\eta^2}\right)\psi'_u - \frac{1}{\sqrt{g}}\psi^*\left(\frac{\partial\sqrt{g}}{\partial v}\right)\left(\frac{\hbar^2}{2v}\right)\psi'_v \quad (2.17)$$

ensures automatically Hamilton's principle, since the arbitrary state functions ψ are assumed to be time independent.

It might be thought that the employment f the conventional calculus of variation could give rise to a quantal equation of motion which is in fact the corresponding Euler–Lagrange differential equation that ensures the minimization of the quantal action. However, the conventional calculus of variation suffers from the fact that the endpoints are restricted to be fixed. In fact this restriction is probed not only to be unnecessary, but also to prohibit the Euler–Lagrange differential operator from being self-adjoint (Embaby, 1978).

We adopt the generalized calculus of variation that developed by Morsy *et al.* (1986). Following a similar procedure to that outlined in Fathia (1980), we can derive the generalized Euler–Lagrange differential equation which satisfies the Hamilton principle given by (2.16), namely

$$\left[\frac{\partial}{\partial\psi^*} + \left(\tilde{\delta}_u - \frac{1}{\sqrt{g}}\frac{\partial}{\partial u}\sqrt{g} - \frac{1}{\sqrt{g}}\left(\frac{\partial\sqrt{g}}{\partial u}\right)\frac{\partial}{\partial\psi'^*_u} + \left(\tilde{\delta}_v - \frac{1}{\sqrt{g}}\frac{\partial}{\partial v}\sqrt{g} - \frac{1}{\sqrt{g}}\left(\frac{\partial\sqrt{g}}{\partial v}\right)\frac{\partial}{\partial\psi'^*_v}\right]F = 0 \quad (2.18)$$

or

$$\left[\frac{\partial}{\partial\psi} + \left(\tilde{\delta}_u - \frac{1}{\sqrt{g}}\frac{\partial}{\partial u}\sqrt{g} - \frac{1}{\sqrt{g}}\left(\frac{\partial\sqrt{g}}{\partial u}\right)\frac{\partial}{\partial\psi'_u} + \left(\tilde{\delta}_v - \frac{1}{\sqrt{g}}\frac{\partial}{\partial v}\sqrt{g} - \frac{1}{\sqrt{g}}\left(\frac{\partial\sqrt{g}}{\partial v}\right)\frac{\partial}{\partial\psi'_v}\right]F = 0 \quad (2.19)$$

in which

$$\tilde{\delta}_u = \delta(u - u_2) - \delta(u - u_1) \tag{2.20a}$$

$$\tilde{\delta}_v = \delta(v - v_2) - \delta(v - v_1) \tag{2.20b}$$

where $\delta(x - x_k)$ denotes the Dirac delta function.

Now, on inserting the explicit expression for the function F as given by (2.17) into the above pair (2.18) and (2.19), we get

$$\left[-\frac{\hbar^2}{2}\frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial u}\sqrt{g}\,\frac{1}{\nu\eta^2}\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\sqrt{g}\,\frac{1}{\nu}\frac{\partial}{\partial v}\right)\right.$$

$$\left. + \left(\frac{\hbar^2}{2\nu\eta^2}\tilde{\delta}_u\frac{\partial}{\partial u} + \frac{\hbar^2}{2\nu}\tilde{\delta}_v\frac{\partial}{\partial v}\right) + V - \lambda\right]\psi(u, v) = 0 \tag{2.21}$$

togehter with its complex conjugate.

Furthermore, and by virtue of the explicit expression for g given by (2.9), we get

$$\left\{-\frac{\hbar^2}{2\nu(u)}\left[\eta^{-2}(v)\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \frac{K}{\eta(v)}\frac{\partial}{\partial v}\right] + \left(\tilde{\delta}_u\frac{\hbar^2}{2\nu(u)\eta^2(v)}\frac{\partial}{\partial u}\right.\right.$$

$$\left.\left. + \tilde{\delta}_v\frac{\hbar^2}{2\nu(u)}\frac{\partial}{\partial v}\right) + v(u, v) - \lambda\right\}\psi(u, v) = 0 \tag{2.22}$$

in which the fission coordinate curvature $K_0\theta(u_0 - |u|)$ is constant in the interaction region $|u| \leq u_0$ and zero outside.

Moreover, the Lagrange multiplier λ, which is assumed to be an arbitrary parameter, plays the role of fission energy E. In this respect the equation of motion given by (2.22) becomes

$$\left[-\frac{\hbar^2}{2\nu\eta^2}(D_u^2 - \tilde{\delta}_u D_u) - \frac{\hbar^2}{2\nu}\left(D_v^2 - \left(\tilde{\delta}_v - \frac{K}{\eta}\right)D_v\right)\right.$$

$$\left. + V(u, v) - E\right]\psi(u, v) = 0 \tag{2.23}$$

in which $V(u, v)$ and $\psi(u, v)$ denote the potential energy and the wave function of the system, respectively.

The equation of motion given by (2.23) differs from that previously established for nuclear fission (Morsy *et al.*, 1979) by the presence of the boundary differential terms:

$$-\frac{\hbar^2}{2\nu\eta^2}\tilde{\delta}_u D_u + \frac{\hbar^2}{2\nu}\tilde{\delta}_v D_v$$

These boundary terms, which are associated with the kinetic energy differential operator, are in fact responsible for restoring its self-adjointness. Such terms generally do not vanish except under some special representations.

3. REDUCTION OF THE EQUATION OF MOTION

In the preceding section, we established a two-dimensional equation of motion for nuclear fission by employing the generalized calculus of variation for minimizing the quantal action in natural collision coordiunates as required by Hamilton's principle.

In the present section, the completeness of a set of vibrational eigenfunctions is exploited to separate out the two-dimensional equation of motion into a vibrational eigenvalue equation and a set of coupled interchannel one-dimensional equations which describe the translational motion. Such set can be decoupled in the framewrok of the optical model approximation. The resulting integrodifferential equation can be reduced to an equivalent differential equation after employing the effective mass approximation.

We expand the potential energv as a Taylor series in powers of the vibrational coordinates v namely

$$V(u, v) = V(u, 0) + \frac{\partial}{\partial v} V(u, v)|_{v=0} v + \frac{1}{2!} \frac{\partial^2}{\partial v^2} V(u, v)|_{v=0} v^2 + \cdots \tag{3.1}$$

However, the second term vanishes by the fission coordinates and consequently (2.23) becomes

$$\left[-\frac{\hbar^2}{2\nu(u)\eta^2(v)} (D_u^2 - \tilde{\delta}_u D_u) - \frac{\hbar^2}{2\nu(u)} \left[D_v^2 - \left(\tilde{\delta}_v - \frac{K}{\eta(v)} \right) D_v \right] + V(u, 0) \right.$$

$$\left. + \frac{1}{2} \nu(u) w^2(u) v^2 - E \right] \psi(u, v) = 0 \tag{3.2}$$

in which $w(u)$ represents the local frequency

$$\nu(u) w^2(u) = \frac{\partial^2}{\partial v^2} V(u, v)|_{v=0} \tag{3.3}$$

The above partial differential equation can be reduced to two uncoupled ordinary differential equations in u and v. To accomplish this, let us assume that there exists a set of vibrational eigenfunctions $\phi_n(u, v)$ which are orthogonal,

$$\int_{-\infty}^{\infty} dv \, \phi_m^*(u, v) \phi_n(u, v) = \delta_{mn} \tag{3.4a}$$

and complete,

$$\sum_{n=0}^{\infty} \phi_m^*(u', v)\phi_n(u, v) = \delta(u - u') \tag{3.4b}$$

Such a complete set satisfies the following eigenvalue equation:

$$\left\{-\frac{\hbar^2}{2\nu(u)}\left[D_v^2 - \left(\tilde{\delta}_v - \frac{K}{\eta(v)}\right)D_v\right] + \frac{1}{2}\nu(u)w^2(u)v^2\right\}\phi_n(u, v)$$
$$= \epsilon_n(u)\phi_n(u, v) \tag{3.5}$$

in which $\epsilon_n(u)$ denotes the associated eigenenergy. In terms of such a complete set, one can expand the scattering wave function $\psi(u, v)$ as a convergent series as

$$\psi(u, v) = \sum_{n=0}^{\infty} F_n(u)\phi_n(u, v) \tag{3.6}$$

where $F_n(u)$ are the corresponding expansion coefficients. Substituting (3.6) into (3.2) and employing (3.5), we get

$$\sum_{n=0}^{\infty}\left[-\frac{\hbar^2}{2\nu(u)}(D_u^2 - \tilde{\delta}_v D_u)V(u, 0) + \epsilon_n(u) - E\right]F_n(u)\phi_n(u, v) = 0 \tag{3.7}$$

On multiplying the lhs of the above equation by η^2, and adding and subtracting the quantity $(V(u, 0) + \epsilon_n(u) - E)$, we obtain

$$\sum_{n=0}^{\infty}\left\{-\frac{\hbar^2}{2\nu(u)}(D_u^2 - \tilde{\delta}_u D_u) + V(u, 0) + \epsilon_n(u) - E + [\eta^2(v) - 1](V(u, 0)\right.$$
$$\left. + \epsilon_n(u) - E\right\}F_n(u)\phi_n(u, v) = 0 \tag{3.8}$$

which after projecting out the vibrational eigenfunction $\phi_n(u, v)$ and using the orthonormality equation (3.4a), reads

$$\sum_{n=0}^{\infty}\left\{-\frac{\hbar^2}{2\nu(u)}L_{mn}(u) - \frac{\hbar^2}{2\nu(u)}M_{mn}(2D_u - \tilde{\delta}_u) + \left[-\frac{\hbar^2}{2\nu(u)}(D_u^2 - \tilde{\delta}_u D_u)\right.\right.$$
$$\left.\left. + V(u, 0) + \epsilon_n(u) - E\right]\delta_{mn} + [V(u, 0) + \epsilon_n(u) - E]N_{mn}(u)\right\}F_n(u) = 0 \tag{3.9}$$

where $D_x(x = u, v)$ denotes d/dx, and $L_{mn}(u)$, $M_{mn}(u)$, and $N_{mn}(u)$ stand for the matrix elements

$$L_{mn}(u) = \int_{-\infty}^{\infty} dv\ \phi_m^*(u, v)\phi_n''(u, v) \tag{3.10}$$

$$M_{mn}(u) = \int_{-\infty}^{\infty} dv\ \phi_m^*(u, v)\phi_n'(u, v) \tag{3.11}$$

$$N_{mn}(u) = \int_{-\infty}^{\infty} dv\ \phi_m^*(u, v)[\eta^2(v) - 1]\phi_n(u, v) \tag{3.12}$$

However, equation (3.9) can be rewritten as a set of equations:

$$\sum_{n=0}^{\infty} \left\{ \delta_{mn} \left[-\frac{\hbar^2}{2v(u)} (D_u^2 - \tilde{\delta}_u D_u) + V(u, 0) + \epsilon_n(u) - E \right] + U_{mn}(u) \right\} F_n(u) = 0 \tag{3.13}$$

where U_{mn} stands for the coupling potential:

$$U_{mn}(u) = -\frac{\hbar^2}{2v(u)} [L_{mn}(u) + M_{mn}(u)(2D_u - \tilde{\delta}_u)] + [V(u, 0) + \epsilon_n(u) - E]N_{mn} \tag{3.14}$$

which directly couples the channel m with the channel n.

The set of coupled channel equations given by (3.13) differs from those previously established by Morsy *et al.* (1979) in the presence of a differential boundary term $[\hbar^2/2v(u)]\tilde{\delta}_u D_i$ in the kinetic energy differential operator, which is responsible for restoring self-adjointness.

Now, the above set of coupled equations can be solved by means of numerical integration techniques due to Marcus (1968) and Conner and Marcus (1970), but such techniques evidently require considerable effort. Alternatively we may decouple such a set of equations by employing the optical approximation of Morsy *et al.* (1979) and Feshbach (1962, 1968). To accomplish this, let us introduce an effective nonlocal optical potential in terms of which equation (3.13) can be written as

$$\left[-\frac{\hbar^2}{2v(u)} (D_u^2 - \tilde{\phi}_u D_u) + V(u, 0) + \epsilon_n(u) - E \right] F_n(u) + \int_{-\infty}^{\infty} du'\ U(u, u')F_n(u') = 0 \tag{3.15}$$

in which the kernel $U(u, u')$ is principally governed by the criterion

$$\sum_{m=0}^{\infty} U_{mn}(u)F_n(u) = \int_{-\infty}^{\infty} du'\ U(u, u')F_b(u') \tag{3.16}$$

This criterion has a certain similarity to that conventionally known as the adiabatic approximation (Levine, 1968), but is more transparent.

Now, our immediate task is to solve the scattering equation (3.15), which involves a nonlocal potential. However, explicit nonlocal potential problems are far less convenient to handle analytically than local ones. Accordingly, we shall convert the above integrodifferential equation into an approximate equivalent differential equation.

This can be accomplished by employing a moment expansion for the nonlocal potential operator. More precisely, we expand the scattering wave function $F_n(u')$ in a Taylor series around u as

$$F_n(u') = \sum_{k=0}^{\infty} \frac{(u' - u)^k}{k!} \frac{\partial^k}{\partial u^k} F_n(u')|_{u'=u} \tag{3.17}$$

by virtue of which the nonlocal potential operator can be expressed as

$$\int_{-\infty}^{\infty} du'\ U(u, u')F_n(u') = \sum_{k=0}^{\infty} U_k(u) \frac{\partial^k}{\partial u^k} F_n(u) \tag{3.18}$$

where $U_k(u)$ stands for

$$U_k(u) = \frac{1}{k!} \int_{-\infty}^{\infty} du'\ U(u, u')(u - u')^k \tag{3.19}$$

At this point, let us assume that the nonlocality of the interaction is not so strong, such that the terms of order higher than $k = 2$ in the above series can be neglected. Substitution of expression (3.18) into (3.15) yields a second-order differential equation with a local potential,

$$\left\{-\left[\frac{\hbar^2}{2\nu(u)} - U_2(u)\right]D_u^2 + \left[\frac{\hbar^2}{2\nu(u)}\tilde{\delta}_u + U_1(u)\right]D_u \right.$$

$$\left. + U_0(u) + V(u, 0) + \epsilon_n(u) - E\right\}F_n(u) = 0 \tag{3.20}$$

in which

$$\int_{-\infty}^{\infty} du'\ U(u, u')(u', u)^k \leq \frac{\epsilon}{l}$$

Moreover, the first-order differential operator can be reduced by inserting the substitution

$$F_n(u) = W(u)\tilde{\phi}_n(u) \tag{3.21}$$

into equation (3.20), then adding and subtracting the quantity $U_2\tilde{\delta}_u$, as a result of which we obtain

$$\left\{-\left(\frac{\hbar^2}{2v(u)} - U_2\right)D_u^2 + \left[\left(\frac{\hbar^2}{2v} - U_2\right)\tilde{\delta}_u + U_1 - \frac{\hbar^2}{v}\frac{w'}{w} + 2\frac{w'}{w}U_2 + U_2\tilde{\delta}_u\right]D_u - \left(\frac{\hbar^2}{2v} - U_2\right)\frac{w''}{w} + \left(\frac{\hbar^2}{2v}\tilde{\delta}_u + U_1\right)\frac{w'}{w} + U_0(u) + V(u, 0) + \epsilon_n(u) - E\right\} \times \tilde{\phi}_n(u) = 0 \tag{3.22}$$

Let us assume that $w(u)$ is chosen to satisfy the following first-order differential equation:

$$2\left(-\frac{\hbar^2}{2v} + U_2\right)w' + (U_1 + U_2\tilde{\delta}_u)w = 0 \tag{3.23}$$

Consequently, equation (3.22) can be expressed as

$$\left\{-\left[\frac{\hbar^2}{2v(u)} - U_2(u)\right](D_u^2 - \tilde{\delta}_u D_u) + U(u) + U_0(u) + V(u, 0) + \epsilon_n(u) - E\right\}\tilde{\phi}_n(u) = 0 \tag{3.24}$$

where

$$U(u) = -\left(\frac{\hbar^2}{2v} - U_2\right)\left[\frac{1}{2}D_u\left(\frac{U_1 + U_2\tilde{\delta}_u}{\hbar^2/2v - U_2}\right) + \frac{1}{4}\left(\frac{U_1 + U_2\tilde{\delta}_u}{\hbar^2/2v - U_2}\right)^2\right] + \frac{1}{2}\left(\frac{\hbar^2}{2v}\tilde{\delta}_u + U_1\right)\frac{U_1 + U_2\tilde{\delta}_u}{\hbar^2/2v - U_2} \tag{3.25}$$

Finally, (3.24) can be expressed simply as

$$\left[-\frac{\hbar^2}{2\tilde{v}(u)}(D_u^2 - \tilde{\delta}_u D_u) + \tilde{U}_n(u) - E\right]\tilde{\phi}_n(u) = 0 \tag{3.26}$$

in which $\tilde{v}(u)$ and $\tilde{U}(u)$ denote, respectively, the effective reduced mass

(ERM) and effective fission barrier (EFB), which are given respectively in the following forms:

$$\tilde{v}(u) = \frac{v(u)}{1 - 2v(u)U_2(u)/\hbar^2} \tag{3.27}$$

and

$$\tilde{U}_n(u) = U(u) + U_0(u) + V(u, 0) + \epsilon_n(u) \tag{3.28}$$

The effect of the interchannel interaction is contained in both the EFB and ERM, which, in general, are complex quantities in view of the fact that the kernel $U(u, u')$ is assumed to be a complex optical potential.

Furthermore, the presence of the first-order differential boundary term $[\hbar^2/2\tilde{v}(u)]\tilde{\delta}_u D_u$ is responsible for making the kinetic energy to be intrinsically self-adjoint.

4. SUMMARY AND CONCLUSIONS

In the present paper, a generalized equation of motion for describing the dynamics of nuclear fission has been established on the basis of Hamilton's principle of least action, after incorporating properly the quantal features and employing the generalized variational calculus. It turns out that our proposed equation of motion is simply the corresponding generalized Euler–Lagrange differential equation, which differs from the conventional Schrödinger equation by the presence of a differential boundary term that is responsible for restoring intrinsic self-adjointness.

The concept of completeness of the variational eigenfunction has been exploited to separate this proposed two-dimensional equation of motion into a vibrational eigenvalue equation and a set of coupled-channel one-dimensional equations. Such coupled-channel equations are intrinsically self-adjoint.

Furthermore, it has been shown that by employing the optical model approximation it is possible to decouple the coupled-channel equation into an integrodifferential equation. Such an equation could be further reduced to an equivalent differential equation in the framework of the effective mass approximation.

This equivalent differential equation has been provided with an effective potential energy barrier, an effective reduced mass that may be complex, and a first-order differential boundary term.

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